Capacity Limits of MIMO Channels with Co-Channel Interference

Matthew Webb, Mark Beach and Andrew Nix
Department of Electrical and Electronic Engineering
University of Bristol, Bristol, BS8 1UB, UK
Email: {M.W.Webb, M.A.Beach, Andy.Nix}@bristol.ac.uk

Abstract— **In this paper we consider both analytically and via Monte Carlo simulation the variation of the information theoretic capacity of multiple-input, multiple output (MIMO) communication systems. We present a way of explicitly examining the effect of interference on the MIMO sub-channel gains. Using this, we derive asymptotic lower bounds on capacity with many interferers and in high interference-to-noise ratio and present simulation results for them.**

We also study the behaviour of the capacity as a function of the number of interferers, the interference-to-noise ratio (INR) and the transmit and receive array sizes. It is found that, in interference, the benefit of providing multiple transmit antennas is very small indeed from a capacity perspective, whilst the number of receive antennas is critical both to the capacity and the number of users the system can mitigate interference from. Unusually, it is seen that, with a modification of the MIMO channel matrix to incorporate interference, higher correlation in the channel yields higher capacity rather than the more familiar desire for decorrelated channels.

I. Introduction

Dual antenna array wireless communication systems are currently a topic of great research interest owing to their promise of dramatic system performance and capacity gains over present technologies. By deploying multiple antennas at both ends of the link, these architectures create a multiple-input, multiple-output (MIMO) channel. Such channels offer the possibility of substantial data throughput gains via the use of space–time layering [1], [2] or improved error performance by exploiting diversity and coding e.g. [3]. Given the increasing demand for high-rate wireless data, in both indoor and outdoor environments, in this paper we will focus on the capacity gains.

Inevitably, in any wireless system a given user will suffer (co-channel) interference from other users. The effects of such interference have been subjected to some limited analysis by other authors in, for example, [4]–[6] who principally assume that the interference is spatially white. In fact, of course, the interference does not emanate equally from all directions — it will have some spatial colour. Recently, this structuring of the interference has begun to receive some attention, beginning with [5] and most notably in [7]–[9].

Future wireless systems will offer the end user multi-rate data services, with these bearers supporting a variety of features such as real-time operation, QoS guarantees, video quality, etc. In general, a user's transmit power will be proportional to their data-rate and so there has been some work carried out recently [7] investigating how the distribution of interference power affects capacity. The work in [7] extends that in [5] by giving a more explicit form to the interference and investigating how the capacity behaves with fixed total interference-plus-noise power and varying number of interferers. Such a constraint is reasonable since there is likely to be some means of controlling the other users in our cell, particularly if they are all subscribed to the same network operator. In [7] it is concluded that MIMO performs with greater spectral efficiency with *few, high-power* users than with *many, low-power* ones but no explanation is given.

The principal contribution of the present work is an approach which shows in more detail than the existing literature the impact of interference on the structure of the MIMO channel matrix as the number or power of the interferers is increased. This allows us to provide a rigorous explanation of the conclusions in [7] and to derive asymptotic lower bounds on the capacity in the presence of many interferers and in high interference-to-noise ratio (INR). We also investigate more generally the capacity of MIMO with varying numbers of interferers and differing transmit and receive array sizes.

Though we arrive at some of the same predictions as [8], we have a rather different setting of numbers of interferers and INR instead of array size. Our new approach offers a more explicit understanding of the manner in which interference impinges on the performance of MIMO systems, regardless of their size.

II. System Model

We consider a single-user, narrowband link with cochannel interference from other users in the same bandwidth transmitting at the same time. Following the analysis in [7] we will assign each interferer an equal fraction of the fixed total interference power as part of our effort to establish whether MIMO provides higher spectral efficiency with few, high power users or with many low power ones. We equip the desired user with n_T transmit antennas and n_R receive antennas. Each interfering user has n_I transmit antennas, but is seen by the same set of n_R receive antennas as the desired user.

Thus, the total received signal may be modelled as:

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \underbrace{\sqrt{\frac{P_I}{Ln_I}}\sum_{i=1}^{L}\mathbf{G}_i\mathbf{s}_i^I + \mathbf{w}}_{\mathbf{n}} \tag{1}$$

where $\mathbf{y}$ and $\mathbf{w}$ are complex n_R-vectors with $\mathbf{w}$ the additive white Gaussian receiver noise which we will assume to have unit power. The complex transmitted signals of the desired user and each interferer are represented by the P_T-power n_T-vector $\mathbf{s}$ and the set of L unit power n_I-vectors $\mathbf{s}_i^I$ respectively. The total interference power P_I is distributed equally among all L interferers. The channel matrices, $\mathbf{H} \in \mathbb{C}^{n_R \times n_T}$ and $\mathbf{G}_i \in \mathbb{C}^{n_R \times n_I}$ of the desired user and interferers respectively as well as all of the signal vectors have i.i.d zero-mean, unit-variance complex Gaussian elements and are themselves mutually independent and quasi-static in time.

III. Channel Capacity

We compute the capacity by utilizing a pre-whitened channel matrix in order to make the interference-plus-noise appear white. It is shown in [7] that the capacity assuming no transmit-side channel knowledge may be computed by defining $\tilde{\mathbf{H}} = \mathbf{R}^{-1/2}\mathbf{H}$ whence

$$C = \log_2 \det\left(\mathbf{I}_{n_R} + \frac{P_T}{\sigma^2 n_T}\tilde{\mathbf{H}}\tilde{\mathbf{H}}^\dagger\right) \tag{2}$$

where $\mathbf{R} = \mathbb{E}\{\mathbf{n}\mathbf{n}^\dagger\}$ is the covariance matrix of the combined interference-plus-noise. In the case of known channel and interference covariance at the transmitter, water-filling may of course be employed on the combined channel matrix, although we do not have space here to present results on this possibility.

Since $\tilde{\mathbf{H}}$ contains the effects of both the desired user's channel and all of the interferers' channels we will study it's structure as the number and power of the interferers changes. We consider the singular value decomposition of both matrices in $\tilde{\mathbf{H}}$ and employ the linear unitary operations at transmit and receive familiar from [10]. Thus we define

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_{n_R} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & \\ 0 & \cdots & & \lambda_{n_R} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^\dagger \\ \vdots \\ \mathbf{u}_{n_R}^\dagger \end{bmatrix} \tag{3}$$

and

$$\mathbf{H} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{n_R} \end{bmatrix} \begin{bmatrix} \gamma_1 & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \gamma_N & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & \cdots & & & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}_1^\dagger \\ \vdots \\ \mathbf{w}_{n_T}^\dagger \end{bmatrix} \tag{4}$$

as the SVDs $\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\dagger$ and $\mathbf{H} = \mathbf{V}\mathbf{\Gamma}\mathbf{W}^\dagger$. We will choose to order the singular values such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_R}$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_N$ with $N = \min(n_T, n_R)$ the rank of the desired user's channel matrix.

Expanding the channel matrix shows that

$$\tilde{\mathbf{H}} = \mathbf{U}\mathbf{V}^{-1/2}\mathbf{U}^\dagger\mathbf{V}\mathbf{\Gamma}\mathbf{W}^\dagger. \tag{5}$$

In [10], in the absence of interference, the familiar result of 'orthogonal spatial subchannels' is derived via linear operations at transmit and receive. In our case, the equivalent of the linear operations would be for the transmitter to left-multiply $\mathbf{s}$ by $\mathbf{W}$ and the receiver to left-multiply the received signal by $\mathbf{U}^\dagger$. Substituting these into (1), noting that we have 'whitened' the interference which is hence absorbed into $\mathbf{w}$, yields the equivalent of [10, Eq. (9)] in interference:

$$\mathbf{y} = \mathbf{\Phi}\mathbf{s} + \mathbf{w} \tag{6}$$

where

$$\mathbf{\Phi} = \mathbf{\Lambda}^{-1/2}\mathbf{U}^\dagger\mathbf{V}\mathbf{\Gamma}. \tag{7}$$

Space restrictions prevent us including the proof, but we can show that the capacity may be computed using $\mathbf{\Phi}$ in place of $\tilde{\mathbf{H}}$ in (2). This representation of the MIMO channel separates the effect of interference from the effect of the desired user's channel and allows to consider them independently. Thus, we will examine the effect on $\mathbf{\Phi}$ of varying the number and power of the interferers.

We remark first that given our assumption of a Gaussian transmission codebook, we can develop on the basis of single-antenna (SIMO) interferers since we may simply group together $n_I > 1$ of them if we want to form a MIMO interferer.

By carrying out the matrix multiplication in (7) it is seen that

$$\mathbf{\Phi} = \left[\begin{array}{ccc|c} \gamma_1\tilde{\lambda}_1(\mathbf{u}_1 \cdot \mathbf{v}_1) & \cdots & \gamma_N\tilde{\lambda}_1(\mathbf{u}_1 \cdot \mathbf{v}_N) & \\ \vdots & \ddots & \vdots & \mathbf{0}_{n_R \times \chi} \\ \gamma_1\tilde{\lambda}_{n_R}(\mathbf{u}_{n_R} \cdot \mathbf{v}_1) & \cdots & \gamma_N\tilde{\lambda}_{n_R}(\mathbf{u}_{n_R} \cdot \mathbf{v}_N) & \end{array}\right] \tag{8}$$

where $\tilde{\lambda}_i \triangleq \lambda_i^{-\frac{1}{2}}$. The partition recognizes the fact that, if $n_R < n_T$, $\mathbf{\Phi}$ will be 'padded' with $\chi = (n_T - n_R)^+$ columns of zeros. It is easily verified that the structure of $\mathbf{R}$ is such that

$$\tilde{\lambda}_i \begin{cases} \leq 1 : 1 \leq i \leq L \\ = 1 : L < i \leq n_R \end{cases}. \tag{9}$$

IV. Bounds on Capacity in Interference

We will now proceed to consider the implications on channel capacity of the structures shown by (8) and (9) and in Section V will simulate the bounds we derive.

A. Bound on capacity with many interferers

Recall from Section II that the interferers' channel matrices are, like the desired user's, circularly symmetric complex Gaussian distributed. Note that $\mathbf{R}$ includes the covariance matrix of the sum of L n_I-dimensional Gaussians. Its off-diagonal elements are non-zero reflecting the colour of the interference. The Central Limit Theorem can be invoked to show that, with many interferers, this covariance matrix

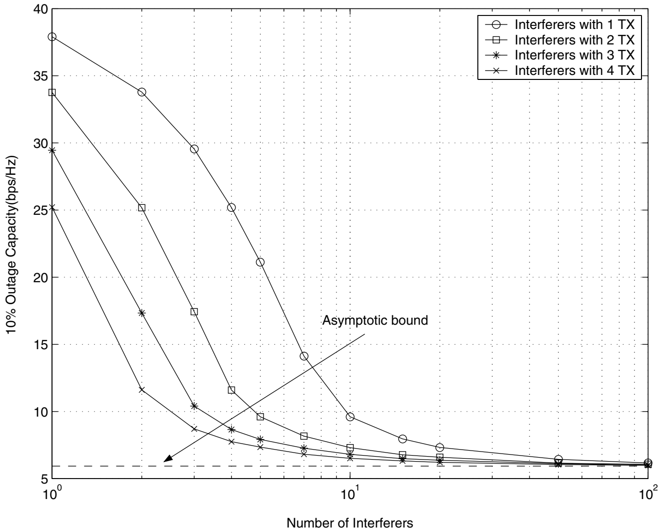

Fig. 1. 10% outage capacity versus number of interferers for MIMO interferers with different numbers of TX antennas. User of interest has 8 TX and 8 RX. Curves shown only for unknown channel and interference.

becomes that of a spatially white Gaussian with all non-diagonal elements zero i.e. as $Ln_I \to \infty$,

$$\mathbf{R} \to \left(\frac{P_I}{\sigma^2}+1\right)\mathbf{I}_N$$
$$\tilde{\mathbf{H}} \to \left(\frac{P_I}{\sigma^2}+1\right)^{-1/2}\mathbf{H}$$
$$C \to \log_2 \det\left[\mathbf{I}_{n_R} + \frac{P_T}{(\sigma^2+P_I)\,n_T}\mathbf{H}\mathbf{H}^\dagger\right] \tag{10}$$

which provides a convenient comparison with (2). The SNR has been replaced by the SINR. This should not be too surprising since the interference now appears spatially white and is subsumed within the thermal noise already present.

B. Bounds on capacity in high INR

In this part, we will examine a bound for the capacity in high INR.

We observe from the definition of $\mathbf{U}$ as a unitary matrix that any increase in INR will result in an increase of the singular values of $\mathbf{R}$. Recalling (9), it is easy to see that, for a given fixed number of interferers, increasing INR will decrease the $\tilde{\lambda}_i$ for $i < L$, but for $i > L$ they will remain fixed at unity. Consider then, the straightforward effect of the reduction of some $\tilde{\lambda}_i$ to zero in the case of $L < n_R$; that is, consider that we have fewer interferers than receive antennas and these interferers transmit with infinite power. Then,

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{0}_{1\times n_R} \\ \vdots \\ \mathbf{0}_{1\times n_R} \\ \phi_{L+1} \\ \vdots \\ \phi_{n_R} \end{bmatrix} \tag{11}$$

where we denote the i^{th} row of (8) as ϕ_i. Thus, the direct manifestation of interference is to reduce the rows of $\mathbf{\Phi}$ controlled by those singular values of $\mathbf{R}$ it affects. In the limit of high INR, these rows' elements vanish. The simplest limit we can formulate is to compute the capacity given by substituting (11) in (2).

Equation (11) allows a further insight. Provided $L < n_R$ the system will be able to support some capacity, regardless of INR. If, though, $L \geq n_R$, the capacity will decline to zero as the INR becomes large. This highlights the effect of degrees-of-freedom in a MIMO system. Intuitively, the MIMO system is able to 'support' as many interferers as it has degrees-of-freedom, and these degrees-of-freedom are set by the number of receiving antennas.

This equation allows one final, straightforward prediction. Obviously, the rows of $\mathbf{\Phi}$ will only be the zero-vector in the limit of high INR, and will reduce toward it as INR increases. This predicts that MIMO systems will perform more efficiently (more bps/Hz) in lower INR regimes. This has been observed by simulation in [7]. More generally, we may say that MIMO systems prefer a noise-dominated regime to an interference dominated one.

We simulate each of the bounds we have derived and examine our predictions in the following section.

V. Simulation Results

By assumption $\mathbf{H}$ and $\mathbf{R}$ are random matrices and so the channel capacity must also be treated as a random variable. We adopt a performance measure of the 10% outage capacity, $C_{0.1}$, where $P(C < C_{0.1}) = 10\%$. In all the following simulations, we used 10 000 instantiations of the channel matrices. For the whole of this section, the signal-to-noise ratio (SNR) and the INR are both 20dB, unless specified otherwise.

A. Bound on capacity with many interferers

We begin by showing how $C_{0.1}$ varies in an 8×8 MIMO system in Fig. 1. Also shown is the asymptotic lower bound predicted by (10). Since the transmitted signals are Gaussian by assumption, we would expect there to be a correspondence between, say, 2 interferers with 1 antenna each and 1 interferer with 2 antennas. This is clearly seen to be the case.

From the plot of the spatially-white interference lower bound, we can see that even with a few interferers it serves as a reasonably accurate prediction of the actual outage capacity.

Fig. 1 also makes clear that MIMO offers greater spectral efficiency when there are a few, high power interferers than when there are many, low power ones even though each of them has proportionally less power. This result was also observed by simulation but not explained in [7] and does warrant explanation. We showed in (11) the effect of adding interferers to be the scaling of individual rows of $\mathbf{\Phi}$, reducing them to zero in the limit of high INR. Adding each interferer reduces another row of $\mathbf{\Phi}$, and also affects those rows already scaled but leaves the others untouched. In high INR, it zeros out an additional row of $\mathbf{\Phi}$. Given the bound we derived in (11), there is a maximum effect an additional interferer (regardless

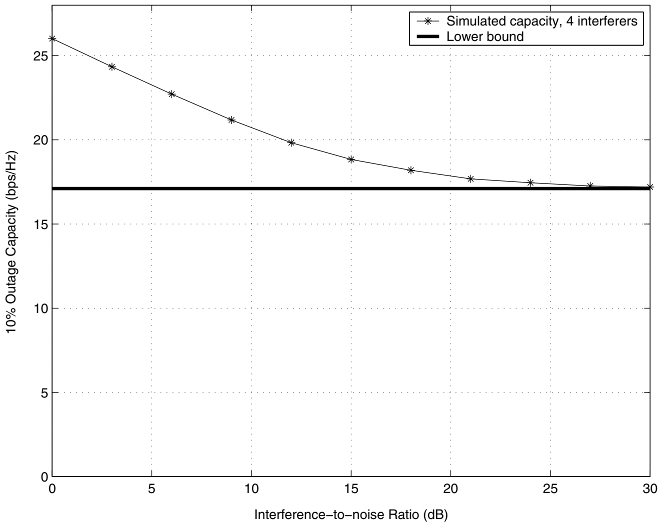

Fig. 2. 10% outage capacity and asymptotic bound versus INR. Desired user has 8 TX and 8RX antennas. 4 interferers, each with 1 TX antenna, SNR=15db. Curves shown only for neither channel nor interference information.

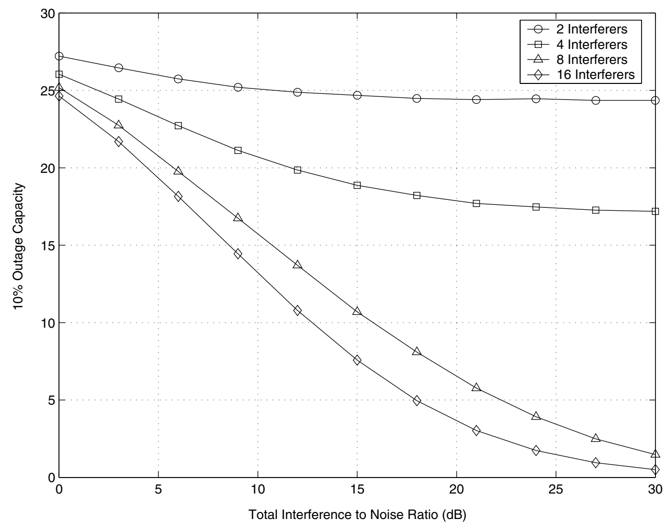

Fig. 3. 10% outage capacity versus INR for an 8×8 system with various numbers of interferers. Each interferer has 1 TX, SNR = 15dB. Curves shown only for neither channel nor interference information.

of its power) can have before $L = n_R$ since it cannot affect rows beyond $i = L$. Hence, having fewer interferers is more desirable, even if they are of higher power. Once $L \geq n_R$, we showed that the capacity would collapse to zero if the INR was high enough. In effect, the capacity for $L \geq n_R$ is 'residual' and available only because of finite INR. If the INR were high enough, the capacity would reach zero as soon as $L = n_R$. This observation explains the saturation observed in Fig. 1 (and in later figures).

Recall that, as L increases $\mathbf{R}$ has decreasing off-diagonal elements; as $\mathbf{R}$ becomes whiter the capacity falls. In $\tilde{\mathbf{H}}$ $\mathbf{R}$ affects the correlations in the channel. The gradual whitening of $\mathbf{R}$ represents a decrease in the additional correlation of the rows/columns of the combined channel. Once $\mathbf{R} = \mathbf{I}$, there is no additional correlation at all. This is unusual — we are accustomed to wanting as little correlation as possible to maximize the capacity but here find that higher correlation achieves that goal. This is explained intuitively by remembering that $\mathbf{R}$ is a covariance matrix and the off-diagonal, cross-correlation elements show the 'predictability' of the interference. The receiver is able to exploit this predictability to mitigate it. As these off-diagonal elements decrease, the interference becomes less and less structured. As an example of the effect of the spatial colour of the interference, consider two rather improbably located interferers who see identical channels to the receiver. Their signals simply sum from the receivers point of view, and they will appear to be just one, higher power, interferer.

B. Bound on capacity in high INR

We turn now to testing the bound in (11) and the associated predictions.

In Fig. 2 we show how the capacity varies with INR for an 8×8 system with 4 interferers. The horizontal line is the '10% outage' of the bound we derived in (11). We see that the bound is approached even at moderate INRs in this case, and serves as a useful prediction of capacity in this case. If there were more interferers, the bound would be approached more slowly since more rows of $\mathbf{\Phi}$ would be affected and it would take longer for all of them to fall to the zero-vector. This graph also clearly supports our prediction that MIMO prefers to operate in a noise-dominated environment than an interference dominated one.

The observation that, provided $L < n_R$, the system will support some theoretic capacity as $\mathbf{\Phi}$ will have some non-zero rows is studied in Fig. 3. We show how the capacity varies with INR for different numbers of interferers. It is readily seen that, whilst there are fewer interferers than receive antennas ($L < n_R$), the capacity tends to an asymptotic limit. This is the theoretic capacity supported by those rows of $\mathbf{\Phi}$ that have not been damaged by interference. On the other hand, once $L \geq n_R$, it is clear that $C_{0.1} \to 0$ as INR $\to \infty$.

C. Effect of transmit and receive array sizes

The previous sections have demonstrated the critical effect of the size of the receive array on the performance of MIMO in interference. We have not yet seen, however, what effect increasing the size of this array will have, other than to expect that the system will support more interferers before the capacity saturates. Hence, in this section we will explore the effect of different size receive and transmit arrays.

Fig. 4 focuses on the effect of changing the number of receiving antennas and Fig. 5 on the effect of changing the number of transmitting antennas. In both these figures, all the interferers have 1 transmit antenna. In Fig. 4 we see the usual result that adding a receive antenna always increases MIMO capacity — it collects more of the transmitted power. We see that adding receive antennas provides better resilience to interference. It raises the curve as expected and, since n_R

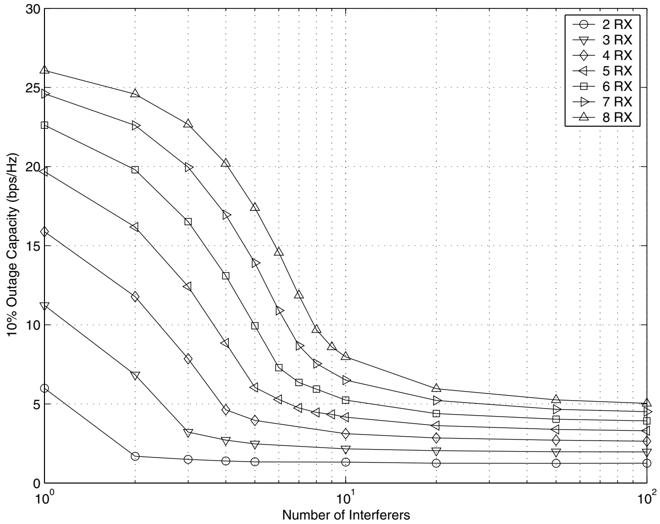

Fig. 4. 10% outage capacity versus number of interferers for different numbers of RX antennas. All interferers have 1TX antenna, desired user has 4TX antennas. Curves shown only for unknown channel and interference.

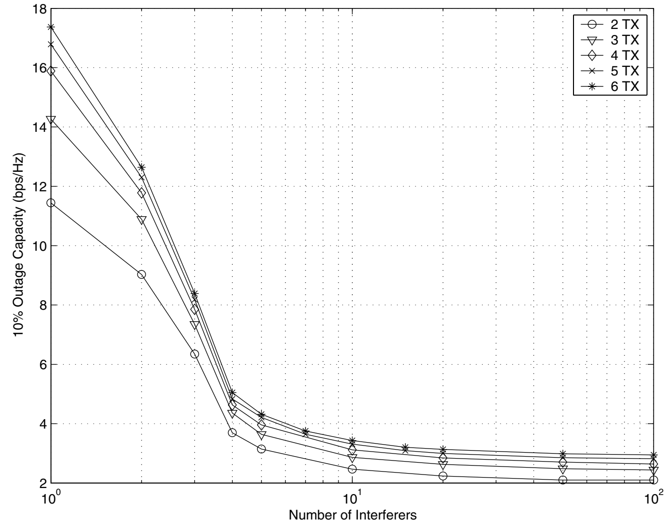

Fig. 5. 10% outage capacity versus number of interferers for different numbers of TX antennas. All interferers have 1TX antenna, desired user has 4RX antennas. Curves shown only for unknown channel and interference.

is now larger and $\mathbf{\Phi}$ thus has more rows, it saturates later as we would expect from (11). Note also that adding receive antennas reduces the bps/Hz cost per interfering antenna. In going from four receive antennas to eight, the loss decreases from about 4bps/Hz/antenna to about 2bps/Hz/antenna.

In Fig. 5, we observe the expected fall in capacity as we remove transmit antennas and the small and diminishing increase as we add them. Without more receive antennas to collect more of the transmitted power (which effectively strengthens the channel's eigenmodes), we would not expect any substantial gain. On the other hand, removing transmit antennas does not decrease the saturation population — we can always support as many interferers as there are receiving antennas before the capacity curve will saturate. As a result, it is worth noting that there is little point providing multiple transmit antennas with large numbers of interferers, since all the curves shown exhibit closely similar capacities beyond the saturation point. There is very little to be gained in this region; by tripling the number of antennas from 2 to 6, the capacity grows only from 2bps/Hz to 3bps/Hz.

VI. Summary

The new matrix investigated here make clearer than before the impact of cochannel interference on MIMO systems. We have used $\mathbf{\Phi}$ to show that adding an interfering antenna has the effect of eliminating one of the well known MIMO subchannels in the limit of high INR whilst leaving the others unchanged. This observation allowed us to derive a lower bound to capacity in high INR. It was also seen that MIMO systems perform more efficiently the more spatially coloured the interference they experience and that, with many interferers, their overall effect is a reduction of the prevailing SNR.

By simulation we found that there was a significant benefit in having $n_R > n_T$, but very little benefit in having $n_T > n_R$ in interference. Indeed, there was little capacity to be gained from providing multiple transmit antennas at all.

VII. Acknowledgments

The authors wish to thank the UK EPSRC and Philips Research Laboratories (PRL) for supporting this work. Discussions with Debbie Raynes and Tim Moulsely from PRL and Chris Williams from the University of Bristol were invaluable.

References

[1] G. J. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas," *Bell Labs. Technical Journal*, pp. 41–59, Oct. 1996.

[2] G. J. Foschini, G. D. Golden, R. A. Valenzuela, and P. W. Wolniansky, "Simplified processing for high spectral efficiency wireless communication employing multi-element arrays," *IEEE J. Select. Areas Commun.*, vol. 17, no. 11, pp. 1841–1852, Nov. 1999.

[3] V. Tarokh, N. Seshadri, and A. Calderbank, "Space–time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, no. 2, pp. 744–765, Mar. 1998.

[4] S. Catreux, P. F. Driessen, and L. J. Greenstein, "Data throughputs using multiple-input multiple-output (MIMO) techniques in a noise- limited cellular environment," *IEEE Trans. Wireless Commun.*, vol. 1, no. 2, pp. 226–235, Apr. 2002.

[5] F. R. Farrokhi, G. J. Foschini, A. Lozano, and R. A. Valenzuela, "Link-optimal space–time processing with multiple transmit and receive antennas," *IEEE Commun. Lett.*, vol. 5, no. 3, pp. 85–87, Mar. 2001.

[6] S. Catreux, P. F. Driessen, and L. J. Greenstein, "Attainable throughput of an interference-limited multiple-input multiple-output (MIMO) cellular system," *IEEE Trans. Commun.*, vol. 49, no. 8, pp. 1307–1311, Aug. 2001.

[7] Y. Song and S. D. Blostein, "MIMO channel capacity in co-channel interference," in *Proc. 21st Biennial Symposium on Communications*, Kingston, Canada, Jan. 2002, pp. 220–224.

[8] A. Lozano and A. M. Tulino, "Capacity of multiple-transmit multiple-receive antenna architectures," *IEEE Trans. Inform. Theory*, vol. 48, no. 12, pp. 3117–3128, Dec. 2002.

[9] R. S. Blum, "MIMO capacity with interference," *IEEE J. Select. Areas Commun.*, vol. 21, no. 5, pp. 793–801, June 2003.

[10] D.-S. Shiu, G. J. Foschini, M. J. Gans, and J. M. Kahn, "Fading correlation and its effect on the capacity of multielement antenna systems," *IEEE Trans. Commun.*, vol. 48, no. 3, pp. 502–513, Jan. 2000.